TRINITY COLLEGE

IN NOMINE DOMINI

Semester Two Examination, 2022

Question/Answer booklet

MATHEMATICS METHODS UNITS 3&4

Section One: Calculator-free

If required by your examination administrator, please place your student identification label in this box

WA student number: In figures

In words

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

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Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	12	12	100	98	65
				Total	100

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4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

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Section One: Calculator-free **35% (50 Marks)**

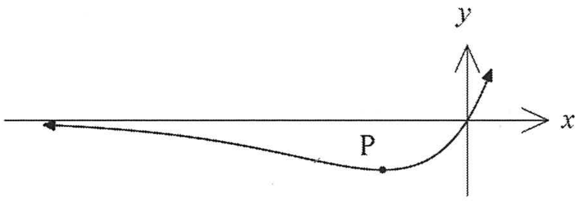
This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(7 marks)**

Let $f(x) = 4xe^{(0.25x-1)}$.

The graph of $y = f(x)$ is shown. It has one stationary point, at P, and one point of inflection.



(a) Clearly show that $f'(x) = (x+4)e^{(0.25x-1)}$. (2 marks)

$u = 4x$ $\quad v = e^{0.25x-1}$

$u' = 4$ $\quad v' = \frac{1}{4}e^{0.25x-1}$

$f'(x) = (4x)(\frac{1}{4}e^{0.25x-1}) + 4e^{0.25x-1}$

$= xe^{0.25x-1} + 4e^{0.25x-1}$

$= (x+4)e^{0.25x-1}$

✓ differentiates exponential

✓ product rule

(b) Determine the coordinates of point P. (2 marks)

P @ $f'(x) = 0$

$(x+4)e^{0.25x-1} = 0$

$x + 4 = 0$

$x = -4$

$f'(-4) = 4(-4)e^{0.25(-4)-1}$

$= -16e^{-2}$

$(-4, \frac{16}{e^2})$

✓ solves $f'(x)=0$

✓ co-ordinates

(c) Determine the values of x for which the curve $y = f(x)$ is concave up. (3 marks)

Need point of inflection

$f''(x) = 0$

$u = x+4$ $\quad v = e^{0.25x-1}$

$u' = 1$ $\quad v' = \frac{1}{4}e^{0.25x-1}$

$f''(x) = \frac{1}{4}(x+4)e^{0.25x-1} + e^{0.25x-1}$

$= (\frac{x+4}{4} + 1)e^{0.25x-1}$

$\frac{x+4}{4} + 1 = 0$

$\frac{x+4}{4} = -1$

$x + 4 = -4$

$x = -8$

$\therefore x > -8$

✓ correct $f''(x)$

✓ x coordinate of p. of inflection

✓ correct inequality for x

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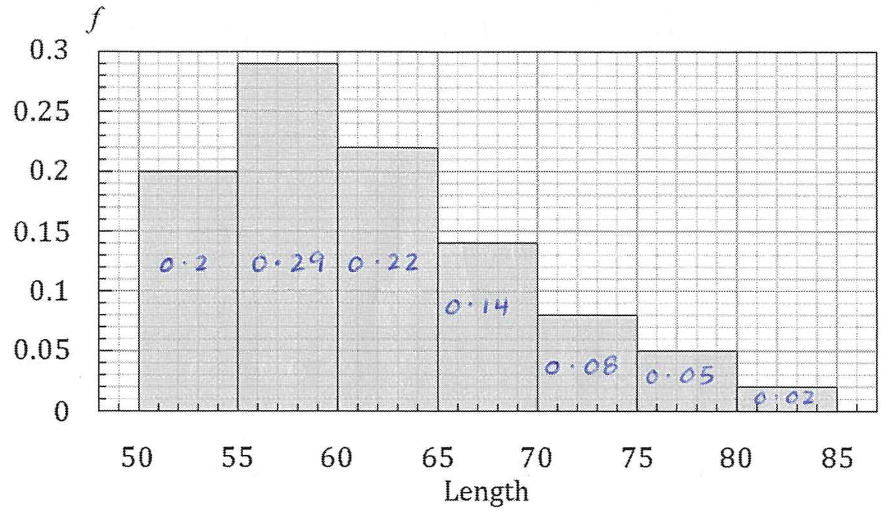
Question 2 **(6 marks)**

The relative frequency histogram below shows the distribution of the lengths in centimetres of a large sample of fish bred in an offshore fish farm.



(a) Use the distribution to determine the probability that

(i) a randomly selected fish will be longer than 70 cm. (1 mark)

$0{\cdot}08 + 0{\cdot}05 + 0{\cdot}02 = 0{\cdot}15$

✓ correct probability

(ii) a randomly selected fish will be exactly 71 cm long. (1 mark)

0

✓ correct probability

(iii) when two fish are randomly selected, one is shorter than 55 cm and the other is not. (2 marks)

$< 55 = 0{\cdot}2$
$> 55 = 0{\cdot}8$

$0{\cdot}2 \times 0{\cdot}8$ OR $0{\cdot}8 \times 0{\cdot}2$

$= 0{\cdot}16 \times 2$

$= 0{\cdot}32$

✓ correct probability for each fish

✓ correct probability

(b) An observer claimed that the distribution of the lengths of fish was approximately normal with a mean of 68 cm and standard deviation of 16 cm. Comment on this claim. (2 marks)

Not bell shaped
Mean of 68 too high
SD too high
∴ Not approx normal

✓ one reasonable comment refers to claim

✓ second comment

Question 3 **(8 marks)**

Determine the following:

(a) $\int 6e^{2x-3}\,dx$. (1 mark)

$= 3e^{2x-3} + c$

✓ correct integral with +c

(b) $\int_0^{\frac{\pi}{8}} \sin(4x)\,dx$. (2 marks)

$= \left[-\frac{1}{4}\cos 4x\right]_0^{\pi/8}$

$= -\frac{1}{4}\cos\frac{\pi}{2} - -\frac{1}{4}\cos 0$

$= 0 - -\frac{1}{4}$

$= \frac{1}{4}$

✓ correct antiderivative

✓ correct value

(c) $f'\left(\frac{\pi}{6}\right)$ when $f(x) = \dfrac{\cos(3x)}{2+\sin(x)}$. (3 marks)

$u = \cos 3x \quad v = 2 + \sin x$

$u' = -3\sin 3x \quad v' = \cos x$

$f'(x) = \dfrac{(2+\sin x)(-3\sin 3x) - \cos x \cos 3x}{(2+\sin x)^2}$

$f'(\pi/6) = \dfrac{(2+\frac{1}{2})(-3) - (\frac{\sqrt{3}}{2})(0)}{(2+\frac{1}{2})^2} = \dfrac{-3(5/2)}{(5/2)^2} = \dfrac{-3}{5/2} = -6/5$

✓ correct quotient rule

✓ correctly differentiates all trig terms

✓ correctly evaluates

(d) $\dfrac{d}{dx}\int_1^x \cos(t+1)\,dt$. (1 mark)

$= \cos(x+1)$

✓ correct result

(e) $\int_0^3 \dfrac{d}{dx}(xe^{2x})\,dx$. (1 mark)

$= 3e^6$

✓ correct result

Question 4 **(7 marks)**

A computer program scans selected text messages passing through a network to see if the message contains a particular keyword. The random variable X takes the value 0 if the keyword is not found, the value 1 if it is found, and has probability distribution

$$P(X=x)=\begin{cases}\dfrac{e^{kx}}{3} & x=0,1\\ 0 & \text{elsewhere.}\end{cases}$$

1 = success : found
0 = fail : not found

(a) Show that the value of the constant k is $\log_e(2)$. (2 marks)

$\frac{e^0}{3}+\frac{e^k}{3}=1$

$\frac{1}{3}+\frac{e^k}{3}=1$

$1+e^k=3$

$e^k=2$

$\ln e^k=\ln 2$

$k=\ln 2$

✓ sub in $x=0$, $x=1$

✓ uses sum of probabilities to find k.

(b) Determine the mean and standard deviation of X. (2 marks)

$\mu = P(X=1) = \frac{e^{(\ln 2)(1)}}{3} = \frac{2}{3}$

$\sigma = \sqrt{p(1-p)} = \sqrt{2/3(1/3)} = \sqrt{2}/3$

✓ correct mean

✓ correct s.d.

(c) Determine the probability that the program finds the keyword in exactly one of the next five randomly selected text messages that it scans. (3 marks)

$n=5$
$p=2/3$

$P(Y=1)=\binom{5}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^4$

$=5\times\frac{2}{3}\times\frac{1}{3^4}$

$=\frac{10}{3^5}$

$=\frac{10}{243}$

✓ correct binomial distribution

✓ correct expression for probability

✓ correct probability

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Question 5 **(8 marks)**

(a) The velocity, v cm per second, of electrically powered model car C at time t seconds is given by $v = \sqrt{2t+3}$. Determine the change in displacement of this car between $t = 0.5$ and $t = 6.5$ seconds. (4 marks)

$v = (2t+3)^{1/2}$

$x = \frac{2}{6}(2t+3)^{3/2} + c$

$x(6.5) - x(0.5) = \frac{2}{6}(16)^{3/2} - \frac{2}{6}(4)^{3/2}$

$= (\frac{2}{6})(4^3) - (\frac{2}{6})(2^3)$

$= \frac{128}{6} - \frac{16}{6}$

$= \frac{56}{3}$ cm

✓ integral for change in displacement

✓ integral

✓ sub in upper & lower bounds

✓ correct displacement

(b) The speed, s cm per second, of model car D at time t seconds is given by $s = e^{\sqrt{2t+3}}$, so that when $t = 3$, its speed was 20.1 cm per second. Use the increments formula to determine a decimal approximation for the speed of this car when $t = 3.05$. (4 marks)

$t = 3$

$\delta t = 0.05$

$s = e^{(2t+3)^{1/2}}$

$u = (2t+3)^{1/2}$

$u' = \frac{1}{2}(2t+3)^{-1/2} \cdot 2$

$= \frac{1}{\sqrt{2t+3}}$

$\frac{ds}{dt} = \frac{1}{\sqrt{2t+3}} e^{\sqrt{2t+3}}$

$\frac{\delta s}{\delta t} = \frac{ds}{dt}$

$\delta s = \delta t \cdot \frac{ds}{dt}\Big|_{t=3}$

$= (0.05)\frac{1}{3}e^{\sqrt{2t+3}}$

$= (0.05)(\frac{1}{3})(20.1)$

$= 0.335$

$\therefore 20.1 + 0.335 = 20.435$ cm/s

✓ $\frac{du}{dt}$

✓ $\frac{ds}{dt}$

✓ increments formula use

✓ speed of car

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Question 6 **(8 marks)**

Components A and B form part of an electronic circuit, and properties of these components are measured t seconds after the circuit is turned on.

(a) The rate of change of temperature, T °C, of component A is given by $\frac{dT}{dt} = \frac{16t}{2t^2 + 5}$. Determine, in simplest form, the increase in temperature of this component during the first 5 seconds. (4 marks)

$$\frac{dT}{dt} = \frac{16t}{2t^2+5}$$

$$\left[4 \ln(2t^2+5)\right]_0^5$$

$$4 \ln 55 - 4 \ln 5$$

$$4 \ln 11 \ C^\circ$$

(accept ln 14641)

✓ integral for change
✓ integrates rate of change
✓ sub in upper & lower
✓ correct increase

(b) The current, I amps, flowing through component B reaches a peak very quickly and then declines as time goes on, as modelled by $I(t) = \frac{3 + \ln(t)}{3t}$. Determine, in simplest form, the maximum current that flows through this component. (4 marks)

$u = 3 + \ln t \quad v = 3t$
$u' = \frac{1}{t} \quad v' = 3$

$$\frac{dI}{dt} = \frac{3t\left(\frac{1}{t}\right) - 3(3 + \ln t)}{9t^2}$$

$$= \frac{3 - 9 - 3\ln t}{9t^2} = 0$$

$$-6 - 3\ln t = 0$$

$$-3 \ln t = 6$$

$$\ln t = -2$$

$$t = e^{-2}$$

$$I(e^{-2}) = \frac{3 + \ln e^{-2}}{3e^{-2}} = \frac{3-2}{3/e^2} = \frac{e^2}{3} \text{ amps}$$

✓ quotient rule
✓ derivative
✓ root of derivative
✓ calculates max in simplified form.

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Question 7 **(8 marks)**

Let $f(x) = k\log_5(x+5) + c$, where k and c are constants.

The graph of $y = f(x)$ intersects line L with equation $4y + 3x + 8 = 0$ when $x = 0$ and $x = -4$.

(a) Determine the value of the constant c and the value of the constant k. (3 marks)

when $x = 0$ $\quad k\log_5 5 + c = -2$

$k + c = -2$

$4y + 8 = 0$

$y = -\frac{8}{4} = -2$

when $x = -4$ $\quad k\log_5 1 + c = 1$

$c = 1$

$4y - 12 + 8 = 0$

$4y = 4$

$y = 1$

$\therefore k = -3$

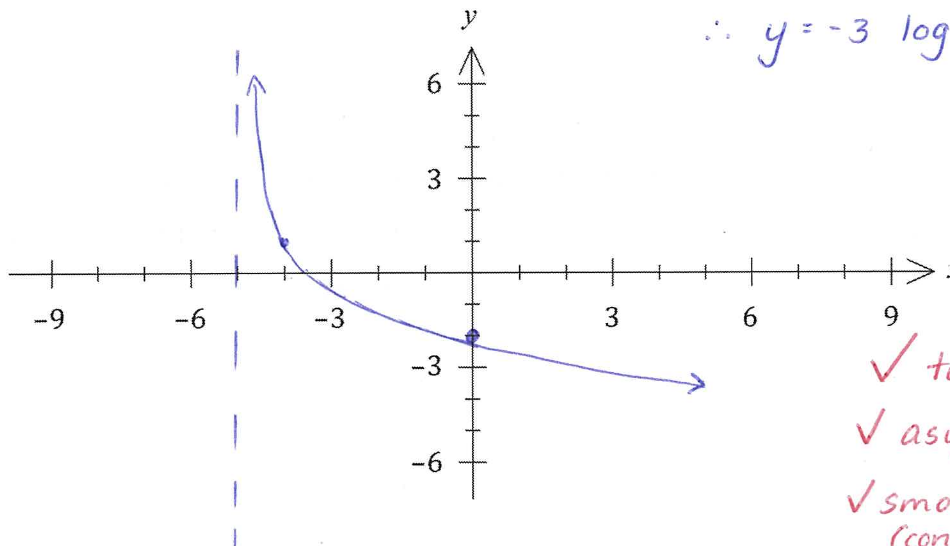
✓ calculates 2 points on curve

✓ c

✓ k

(b) Sketch the graph of $y = f(x)$ on the axes below. (3 marks)

$\therefore y = -3\log_5(x+5) + 1$



✓ two points

✓ asymptote

✓ smooth curve (concave up)

(c) Given that $\log_5(x+5) = \dfrac{\ln(x+5)}{\ln(5)}$, determine the value of x where the slopes of $y = f(x)$ and line L are the same. (2 marks)

$f(x) = -3\log_5(x+5) + 1$

change of base $\quad = -3\dfrac{\ln(x+5)}{\ln 5} + 1$

$= \dfrac{-3}{\ln 5}\ln(x+5) + 1$

$f'(x) = \dfrac{-3}{\ln 5} \cdot \dfrac{1}{x+5} = -\dfrac{3}{4}$

$4y + 3x + 8 = 0$

$4y = -3x - 8$

$y = \frac{-3}{4}x - 2$

$\therefore \ln 5(x+5) = 4$

$x + 5 = \frac{4}{\ln 5}$

$x = \frac{4}{\ln 5} - 5$

✓ differentiate f

✓ x coordinate

End of questions

Supplementary page

Question number: _________

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Semester Two Examination, 2022

Question/Answer booklet

MATHEMATICS METHODS UNITS 3&4

Section Two: Calculator-assumed

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In words

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Section Two: Calculator-assumed **65% (98 Marks)**

This section has **twelve** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

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Question 8 **(7 marks)**

Naltrexone is useful in managing heroin-dependent patients who find it difficult to shift away from dependent use patterns. The blood naltrexone level N of a patient who has received a naltrexone implant was observed to halve every 33 days, from an initial level of 7.7 ng/ml. The level can be modelled by an equation of the form $N = ae^{kt}$, where t is the time in days since the implant was received.

(a) State the value of the constant a and then determine the value of the constant k. (3 marks)

$a = 7.7$

$N = 7.7e^{kt}$

$\frac{1}{2} = e^{33k}$

$k = -0.0210$

$\left(-\frac{\ln 2}{33}\right)$

✓ a
✓ equation using half life
✓ k

The treatment is effective whilst the naltrexone level remains above 1.6 ng/ml.

(b) Determine the number of days that the implant will be effective. (2 marks)

$1.6 = 7.7e^{-0.021t}$

$t = 74.8$

75 days

✓ equation = 1.6
✓ days

(c) Determine the rate at which the naltrexone level is decreasing 15 days after the implant is received. (2 marks)

$\left.\frac{dN}{dt}\right|_{t=15} = -0.021(7.7)e^{-0.021t}$

$= -0.118$

∴ decreasing at a rate of 0.118 ng/ml /day

✓ correct method
✓ rate of decrease

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Question 9 **(8 marks)**

The launch speed of a small projectile fired from a catapult was measured and found to be normally distributed with a mean of 17.4 ms^{-1} and a standard deviation of 0.21 ms^{-1}.

(a) Determine the probability that the projectile is launched with a speed less than 17 ms^{-1}. (1 mark)

$X \sim N(17.4, 0.21^2)$ $P(X < 17) = 0.0284$

✓probability

(b) Determine the probability that the projectile is launched with a speed less than 17.5 ms^{-1} given that its launch speed exceeds 17 ms^{-1}. (2 marks)

$\dfrac{P(17 \le X \le 17.5)}{P(X \ge 17)} = 0.6738$ $\left(\dfrac{0.655}{0.972}\right)$

✓indicates both probabilities

✓correct probability

(c) In a series of 15 launches, determine the probability that the speed of the projectile is less than 17 ms^{-1} in at least 2 of these launches. (2 marks)

$Y \sim B(15, 0.0284)$

$P(Y \geq 2) = 0.0663$

✓binomial dist. with parameters

✓correct probability

(d) The projectile is expected to have a speed exceeding v ms once in every 80 launches. Determine the value of v. (1 mark)

$P(X > v) = \frac{1}{80}$

$v = 17.87\ ms^{-1}$

✓speed (at least 2dp)

(e) The instrument used to measure the launch speed was suspected to underestimate the speed of the projectile by 0.04 ms^{-1}. If this was the case, state the true mean and standard deviation of the distribution of launch speeds for the projectile. (2 marks)

true mean $= 17.4 + 0.04 = 17.44\ ms^{-1}$

true $\sigma = 0.21\ ms^{-1}$

✓correct mean

✓correct s.d.

Question 10 **(6 marks)**

The owners of a shopping mall wanted to confirm their estimate that 35% of local school students visited their mall at least once a week. The owners considered the following three ways of selecting a sample:

A Ask students who turn up to the mall after school.

B Create an online survey and publish a link to it in the local newspaper.

(a) Briefly discuss a source of bias in each sampling method and suggest a better sampling procedure. (3 marks)

A : non response - students might not want to divulge info
A : undercoverge - won't sample students who don't visit mall
A : convenience - only sample students who visit mall

B : Undercoverage - will not sample students who don't see link
B : Self selection - only sample students who volunteer to take survey.

✓ bias in A
✓ bias in B
✓ random sampling

(b) It was found that 105 out of a random sample of 375 students visited the mall at least once a week. Determine the 95% confidence interval for the proportion based on this data and use it to comment on the owner's estimate. (3 marks)

95% $z = 1.96$

$n = 375$

$p = \frac{105}{375} = 0.28$

$0.28 \pm \sqrt{\frac{0.28(1-0.28)}{375}}$

$(0.2346,\ 0.3254)$

95% CI does not contain owner's estimate of 0.35 suggests true value of proportion is likely to be less than 35%.

✓method for CI
✓correct CI (>2dp)
✓uses CI to dispute owner's estimate

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Question 11 **(8 marks)**

(a) A polynomial function is defined by $f(x) = (kx - 1)^3$, where k is a constant. The area under the curve $y = f(x)$ between $x = 3$ and $x = 9$ is 12 square units.

Determine the area under the curve $y = f(x)$ between $x = 3$ and $x = 6$. (4 marks)

$$\int_3^9 (kx-1)^3 = 12 = \left[\frac{1}{4k}(kx-1)^4\right]_3^9$$

$$12 = \frac{1}{4k}(9k-1)^4 - \frac{1}{4k}(3k-1)^4 = 12$$

$$(9k-1)^4 - (3k-1)^4 = 48k$$

$$k = 1/3$$

$$\int_3^6 \left(\tfrac{1}{3}x - 1\right)^3 dx = 3/4 \text{ units}^2$$

✓ integral for area
✓ equation with k
✓ k
✓ area

(b) The graph of another polynomial $y = g(x)$ has a point of inflection at $(3, 7)$ and a stationary point when $x = -1$.

If $g'(x) = 3x^2 + ax + b$, where a and b are constants, determine $g(x)$. (4 marks)

$g'(x) = 0$ @ $x = -1$
$g''(x) = 0$ @ $x = 3$

$0 = 3 - a + b$
$0 = 3 + 18 + b$
$b = -21$

$0 = 6x + a$
$0 = 18 + a$
$a = -18$

$g'(x) = 3x^2 - 18x - 21$
$g(x) = x^3 - 9x^2 - 21x + c$
$(3,7)$ $7 = 27 - 81 - 63 + c$
$c = 124$

$\therefore g(x) = x^3 - 9x^2 - 21x + 124$

✓ a
✓ b
✓ antiderivative of g'(x)
✓ g(x) with c

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Question 12 **(10 marks)**

An online retailer of auto parts knows that on average, 18.5% of parts sold will be returned.

(a) Let the random variable X be the number of parts returned when a batch of 88 parts are sold.

(i) Describe the distribution of X. (2 marks)

Binomial

$X \sim B(88, 0.185)$

✓ Binomial
✓ parameters

(ii) Determine the probability that less than 15% of the parts sold in this batch will be returned. (2 marks)

$0.15 \times 88 = 13.2$

$P(X \leq 13) = 0.2264$

✓ correct binomial probability
✓ correct probability

The retailer takes a large number of random samples of 150 parts from its sales data and records the proportion $\hat{p}$ of returned parts in each sample. Under certain circumstances, the distribution of $\hat{p}$ will approximate normality.

(b) Explain why the retailer can expect the distribution of $\hat{p}$ to closely approximate normality in this case. (3 marks)

$n > 30$

sampling is random

$np \geq 10$ $\quad n(1-p) \geq 10$

states:
✓ samples are random
✓ sample size is large
✓ least number of successes & failures required

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(c) State the parameters of the normal distribution that $\hat{p}$ approximates and use this distribution to determine the probability that the proportion of returns in a random sample of 150 parts is less than 15%. (3 marks)

$\hat{p} = 0.185 \qquad \sigma = 0.0317$

$X \sim N(0.185,\ 0.0317^2)$

$P(\hat{p} < 0.15) = 0.01348$

✓ mean
✓ s.d OR variance
✓ probability

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Question 13 **(8 marks)**

Brass ingots are cast by a metal recycling machine with masses of X kg, where X is a continuous random variable with cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 3 \\ ax^2 - bx & 3 \le x \le 4 \\ 1 & x > 4 \end{cases}$$

(a) Deduce from the cumulative distribution function that the values of the constants a and b are $a = 0.25$ and $b = 0.75$. (3 marks)

$9a - 3b = 0$
$16a - 4b = 1$

$\therefore\ a = 0.25$
$b = 0.75$

✓ lower bound eqn 1
✓ upper bound eqn 2
✓ a and b

(b) Determine the probability that a randomly selected ingot cast by the machine has a mass less than 3.8 kg. (1 mark)

$F(3.8) = \frac{19}{25} = 0.76$

✓ probability

(c) Determine the mean and standard deviation of the masses of ingots cast by the machine. (4 marks)

$F(x) = \frac{1}{4}x^2 - \frac{3}{4}x$ $\quad f(x) = \frac{1}{2}x - \frac{3}{4}$

$E(x) = \int_3^4 x\, f(x)\, dx = 3.5416$ kg

$Var(x) = \int_3^4 \left(x - \frac{85}{24}\right)^2 f(x)\, dx = 0.0816$

$\sigma_x = \sqrt{0.0816} = 0.2857$

✓ p.d.f
✓ mean
✓ integral for variance
✓ s.d.

See next page

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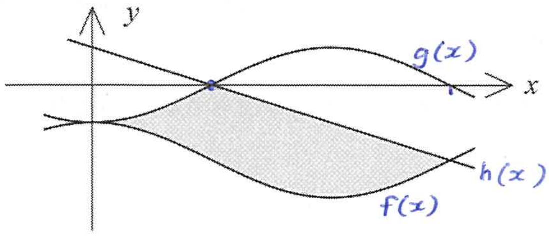
Question 14 **(7 marks)**

Functions f, g and h are defined by

$$f(x) = 10\cos\left(\frac{\pi x}{5}\right) - 20$$
$$g(x) = -10\cos\left(\frac{\pi x}{5}\right)$$
$$h(x) = 10 - 4x.$$

The graphs of these functions are shown to the right.



(a) Determine the area between $y = f(x)$, the x-axis, $x = 3.75$ and $x = 5$. (3 marks)

$$\int_{3.75}^{5} f(x)\,dx$$

$$= -\frac{25\sqrt{2}}{\pi} - 25$$

$$\text{Area} = \frac{25\sqrt{2}}{\pi} + 25 \approx 36.3 \text{ units}^2$$

✓ integral (can have −)
✓ evaluates integral
✓ area (deals with −)

(b) Determine the area of the shaded region enclosed by the three functions. (4 marks)

$10 - 4x = 0$
$4x = 10$
$x = 2.5$

$f(x) = g(x)$
$x = 7.5$

$$A = \int_{0}^{2.5} g(x) - f(x)\,dx + \int_{2.5}^{7.5} h(x) - f(x)\,dx$$

$$= \left(50 - \frac{100}{\pi}\right) + \left(50 + \frac{100}{\pi}\right)$$

$$= 100 \text{ sq units}$$

✓ integral 0 − 2.5
✓ evaluates first integral
✓ integral 2.5 − 7.5
✓ evaluates 2nd integral and Area.

Question 15 **(9 marks)**

In a random sample of 125 adult male Australians, 35 were born overseas. This data is to be used to construct a 90% confidence interval for the proportion of adult male Australians born overseas.

(a) Determine the margin of error for the 90% confidence interval. (3 marks)

$p = \frac{35}{125}$

$\sigma = 0.0402$

$z = 1.645$

$E = 0.0661$

✓ correct p
✓ correct s.d
✓ correct E

(b) State the 90% confidence interval. (1 mark)

$(0.2139, 0.3461)$

✓ correct CI

(c) If 5 similar samples are taken and each used to construct a 90% confidence interval, determine the probability that at least 4 of the intervals will contain the true proportion of adult male Australians who were born overseas. (2 marks)

$X \sim B(5, 0.9)$

$P(X \geq 4) = 0.9185$

✓ correct binomial with parameters

✓ correct probability

(d) The 90% confidence interval for the proportion of adult female Australians born overseas constructed from another random sample was $(0.202, 0.298)$. Determine the number of adult females who were born overseas in this sample. (3 marks)

$E = \frac{0.298 - 0.202}{2}$

$= 0.048$

$p = 0.202 + 0.048$

$= 0.25$

$n = 220$

$\therefore X = 220 \times 0.25 = 55$ females

✓ p and E
✓ n
✓ females

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Question 16 **(9 marks)**

The number of points awarded each time an online game is played is the random variable X, where $E(X) = 3.9$ and X has the following probability distribution.

x	0	1	3	6	c
$P(X = x)$	k	0.25	0.35	0.25	0.10

(a) Determine the value of the constant c and the value of the constant k. (3 marks)

$k + 0.25 + 0.35 + 0.25 + 0.1 = 1$

$k = 0.05$

$0 + 0.25 + 3(0.35) + 6(0.25) + c(0.1) = 3.9$

$c = 11$

✓ k
✓ E(X)
✓ c

(b) Calculate the variance of Y, where $Y = 10X + 5$. (3 marks)

$Var(X) = 0.05(0-3.9)^2 + 0.25(1-3.9)^2 + 0.35(3-3.9)^2 + 0.25(6-3.9)^2 + 0.1(11-3.9)^2$

$= 9.29$

$Var(Y) = 10^2(9.29) = 929$

✓ method for variance
✓ var(X)
✓ var(Y)

When playing a set of 8 games, the points awarded in each game is independent of other games and a player wins a prize if the total number of points scored in the set is at least 35.

(c) A player has completed 6 games in a set and has been awarded a total of 23 points. Determine the probability that they win a prize on completion of the set. (3 marks)

_ _ _ _ _ _ (23) | _ _ (12) | ≥ 35 points

11, 1 / 1, 11: $(0.1)(0.25) \times 2 = 0.05$

11, 3 / 3, 11: $(0.1)(0.35) \times 2 = 0.07$

11, 6 / 6, 11: $(0.1)(0.25) \times 2 = 0.05$

11, 11: $(0.1)(0.1) = 0.01$

6, 6: $(0.25)(0.25) = 0.0625$

$= 0.2425$

✓ all combinations
✓ correct probability of 2 combinations
✓ correct probability

Question 17 **(11 marks)**

The turbidity index I (a measure of purity) of water being treated in tank A can be modelled by the relationship $I = 20e^{-0.25t}$, where t is the time in hours since treatment began.

(a) Express this relationship in the form $t = p\log_e(kI)$, where p and k are constants. (2 marks)

$\frac{1}{20} I = e^{-0.25t}$

$\ln \frac{1}{20} I = -0.25t$

$t = -4 \ln \frac{1}{20} I$

✓exponential to ln
✓simplifies

(b) Determine the time taken, to the nearest minute, for the turbidity index of the water in tank A to halve. (2 marks)

$I = 10$

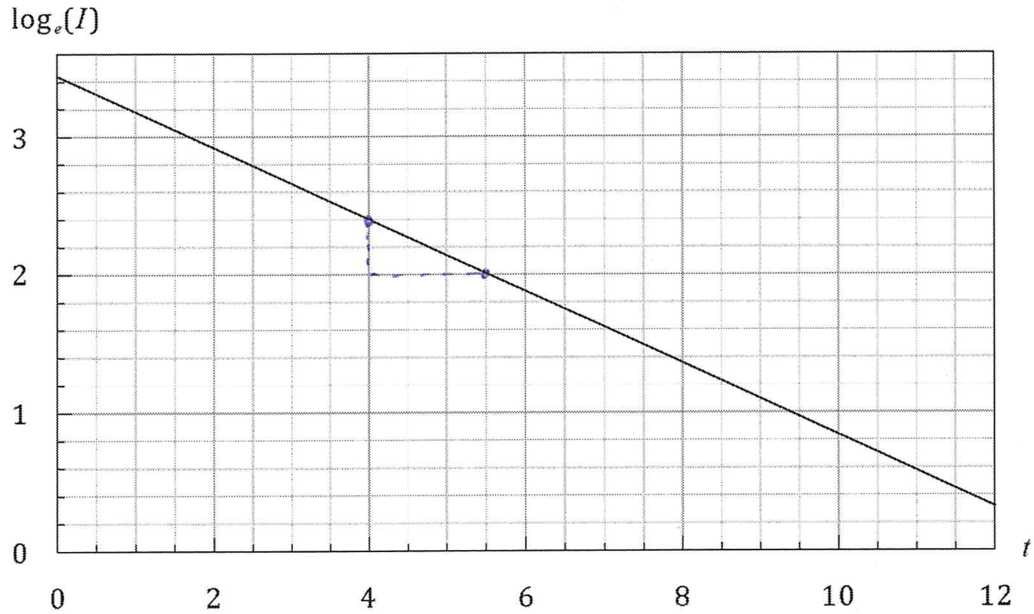
$t = -4 \ln {}^{10}/_{20}$

$t = 2.7726$

2 hr 46 min

✓expression for t
✓ t

Readings of water being treated in tank B were used to construct the graph below, where a linear relationship between $\log_e(I)$ and time t exists. The line passes through the points $(4, 2.4)$ and $(9, 1.1)$.



(c) Determine the turbidity index of the water in tank B when $t = 4$. (1 mark)

$2.4 = \log_e I$

$I = e^{2.4} = 11$

✓ I

(d) Determine the equation of the linear relationship shown in the graph in the form $\log_e(I) = at + b$, where a and b are constants and hence express the turbidity index I as a function of time t for the water being treated in tank B. (3 marks)

$m = \frac{2.4 - 1.1}{4 - 9} = \frac{1.3}{-5} = -0.26$

$\log_e I = -0.26t + 3.44$

$I = e^{-0.26t + 3.44}$

✓ slope & intercept
✓ eqn for $\log_e I$
✓ function for I

Treatment began at 1:00 pm in tank B, and at 2:30 pm in tank A.

(e) Determine the time at which the turbidity indices of the water in the tanks first become the same. (3 marks)

tank A = tank B + 1.5

at $t = 0$ $t = t + 1.5$

$20e^{-0.25t} = e^{3.44 - 0.26(t + 1.5)}$

$t = 5\text{h } 26\text{m}$

$2.30 + 5\text{h } 26\text{m}$

$= 7\text{hr } 56 \text{ pm}$

✓ equation for t
✓ t
✓ time of day

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Question 18 **(8 marks)**

A small body moves along the x-axis with acceleration t seconds after leaving the origin given by $a(t) = 4.32 + kt$ cm/s^2, where k is a constant. The initial velocity of the body is -5 cm/s, and its change in displacement during the fourth second is 7.9 cm.

(a) Determine the maximum velocity of the body. (6 marks)

$t=0 \quad v=-5$

$t=3 \rightarrow t=4$

$x = 7.9$

$v(t) = 4.32t + \frac{kt^2}{2} + c$

$-5 = c$

$x(t) = \left[2.16t^2 + \frac{kt^3}{6} - 5t\right]_3^4 = 7.9$

$k = -9/25$

Max v when $a(t) = 0$

$4.32 - \frac{9}{25}t = 0$

$t = 12$

Max $v(12) = 20.92$ cm/s

✓ expression for velocity
✓ integral for displacement
✓ $x(t)$
✓ k
✓ t
✓ Max V

(b) Determine, to the nearest centimetre, the distance travelled by the body between $t = 0$ and the instant it reaches its maximum velocity. (2 marks)

$\int_0^{12} 4.32t - \frac{9}{50}t^2 - 5 = 153$ cm

✓ method for distance
✓ distance

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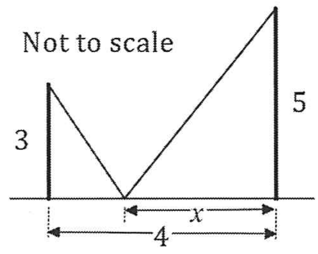
Question 19 **(7 marks)**

Two thin vertical posts, one 5 m and the other 3 m tall, stand 4 m apart on horizontal ground. A small stake is positioned directly between the bases of the posts at a distance of x m from the base of the taller post.



A length of thin wire runs in a straight line from the top of one post, to the stake, and then to the top of the other post.

(a) Calculate the length of the wire when the stake is positioned midway between the bases. (1 mark)

$$L = \sqrt{5^2 + 2^2} + \sqrt{3^2 + 2^2}$$
$$= \sqrt{29} + \sqrt{13}$$
$$\approx 8.99 \text{ m}$$

(exact or > 2dp)
✓ correct length

(b) Use a calculus method to determine where the stake should be positioned to minimise the length of wire, state what this minimum length is and justify that the length is a minimum. (6 marks)

$$L = \sqrt{5^2 + x^2} + \sqrt{3^2 + (4-x)^2}$$

$$\frac{dL}{dx} = \frac{1}{2}\frac{2x}{\sqrt{25+x^2}} + \frac{1}{2}\frac{2(4-x)(-1)}{\sqrt{9+(4-x)^2}}$$

$$= \frac{x}{\sqrt{25+x^2}} + \frac{(x-4)}{\sqrt{9+(4-x)^2}}$$

$\frac{dL}{dx} = 0$ when $x = \frac{5}{2}$ m (2.5 m)

$L(2.5) = 4\sqrt{5} \approx 8.944$ m

$L'(2.4) = -ve$
$L'(2.6) = +ve$ $\therefore$ min OR $L''(2.5) = +ve$ $\therefore$ min

✓ expression for length
✓ first derivative (any for[...]
✓ derivative = 0 and x
✓ min length (exact or 2d[...]
✓ second derivative or sign test
✓ justifies min.

End of questions

Supplementary page

Question number: _________

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Supplementary page

Question number: _________

Supplementary page

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Section Two: Calculator-assumed 65% (98 Marks)

This section has **twelve** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

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Section One: Calculator-free **35% (50 Marks)**

This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.
